Matching of the Shape Function

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Abstract

The shape function $f(k_+)$ describes Fermi motion effects in inclusive semi-leptonic decays such as $B \to X_u + e + \nu$ near the end-point of the lepton spectrum. We compute the leading one-loop corrections to the shape function $f(k_+)$ in the effective theory with a hard cut-off regularization. The matching constant onto full QCD is infrared safe, i.e. the leading infrared singularity represented by the term $\log^2 k_+$ cancels in the difference of integrals. We compare the hard cut-off result with the result in dimensional regularization, the latter containing an additional factor of two in the coefficient of the $\log^2 k_+$ term, and consequently requiring an oversubtraction.

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1 Introduction

The shape function $f(k_+)$ (see Eq. (2) for the definition) is introduced to describe inclusive semi-leptonic decays such as

$$B \to X_u + e + \nu \tag{1}$$

in the kinematical region with a large jet energy

$$E_X^2 \sim O(m_B^2)$$

and with an "intermediate" invariant mass

$$M_X^2 \sim O(\Lambda_{QCD}\, m_B).$$

This region is close to the end-point of the lepton spectrum, where the background decay $B \to X_c l\nu$ is kinematically forbidden; therefore this is the interesting one for the determination of the Cabibbo-Kobayashi-Maskawa matrix element V_{ub} [1].

The aim of this note is to discuss the renormalization properties of the shape function $f(k_+)$ and to give a complete description of the matching onto full QCD. In order to be meaningful, an effective theory must have the same infrared behaviour of the original, high energy theory. By computing the shape function in the double logarithmic approximation with a hard cut-off, we find indeed the same double logarithm of k_+ as in QCD. In other words, the leading IR singularity, $\log^2 k_+$, cancels in the matching constant (coefficient function), implying the factorization of infrared physics into the shape function.

We compare the hard cut-off result with the result of dimensional regularization. In the latter, it appears an additional factor of two in the term proportional to $\log^2 k_+$, implying that the double logarithm of k_+ does not cancel in the matching constant. This is in contrast to naive expectations. We show, however, that it does exist a non-minimal subtraction scheme where the matching is consistent and the leading infrared singularity cancels. In this scheme we are forced to include terms containing simple logarithms of k_+ (see Eq. (19)), which is not worrysome since in the effective theory k_+ is just an index, labelling the operators defining the shape function[2, 3].

2   The shape function

The shape function is defined as [4]

$$f(k_+) \equiv \langle B(v) \mid h_v^\dagger \delta(k_+ - iD_+)h_v \mid B(v)\rangle \qquad (2)$$

where h_v is the field in the HQET with velocity v ($v^2 = 1$) representing the heavy quark, $D_+ \equiv n \cdot D$ and $n_\mu \equiv (1, 0, 0, -1)$ ($n^2 = 0$). We are considering an effective field theory (EFT) where the heavy b quark is taken in the HQET and the light quark is taken in the LEET [5].

The argument of $f(k_+)$ is related to the kinematics of the process by

$$k_+ = -\frac{Q^2}{2\,v \cdot Q}, \qquad (3)$$

where $Q \equiv m_B v - q$ and q is the momentum of the probe, i.e. of the virtual W boson in the decay (1).

In the semileptonic decay (1), all the hadronic information is contained in the tensor

$$W_{\mu\nu}(v, Q) \equiv \sum_X \langle B|J_\mu^\dagger(0)|X\rangle\langle X|J_\nu(0)|B\rangle\,\delta^4(p_B - q - p_X),$$

where the current is defined as $J_\mu(x) \equiv \bar{u}(x)\Gamma_\mu b(x)$[1]. This tensor is proportional to the absorptive part of the forward scattering tensor

$$T_{\mu\nu}(v, Q) \equiv -i\int d^4x\, e^{-iqx}\langle B|T\left(J_\mu^\dagger(x)J_\nu(0)\right)|B\rangle$$

according to the optical theorem

$$W_{\mu\nu} = -\frac{1}{\pi}\mathrm{Im}\,T_{\mu\nu}. \qquad (4)$$

In EFT, it is well known that $W_{\mu\nu}$ can be expressed in terms of the shape function as

$$W_{\mu\nu}^{EFT} = s_{\mu\nu}\,\frac{1}{2v \cdot Q}\; f(k_+) \qquad (5)$$

where

$$s_{\mu\nu} \equiv \frac{1}{2}\,\mathrm{Tr}\,\left(\Gamma_\mu^\dagger \hat{Q}\Gamma_\nu P_+\right)$$

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[1] In the Standard Model, as usual, $\Gamma_\mu = \gamma_\mu(1 - \gamma_5)$.

is a factor containing the spin effects and $P_+ \equiv (1 + \hat{v})/2$ is the projector on the states with velocity v and positive energy. The term $1/2v \cdot Q$ is the jacobian factor in the change of variable from Q^2 to k_+, as it stems from (3).

We find useful to introduce a light-cone function $F(k_+)$

$$F(k_+) \equiv \langle B(v) \mid h_v^\dagger \frac{1}{iD_+ - k_+ + i\eta} h_v \mid B(v)\rangle \tag{6}$$

such that the shape function is proportional to the absorptive part of $F(k_+)$ via a relation analogous to (4)

$$f(k_+) = -\frac{1}{\pi} \mathrm{Im}\ F(k_+). \tag{7}$$

By using Eq.(6), we can write the tensor $T_{\mu\nu}$ in the EFT in a form resembling Eq.(5)

$$T_{\mu\nu}^{EFT} = s_{\mu\nu}\ \frac{1}{2v \cdot Q}\ F(k_+). \tag{8}$$

Let us observe that a shape function $f(k_+)^{QCD}$ and a light-cone function $F(k_+)^{QCD}$ can be also defined in full QCD by means of the relations:

$$T_{\mu\nu}^{QCD} \equiv (s_{\mu\nu} + \Delta s_{\mu\nu}) \frac{1}{2v \cdot Q}\ F(k_+)^{QCD}$$

and

$$W_{\mu\nu}^{QCD} \equiv (s_{\mu\nu} + \Delta s_{\mu\nu}) \frac{1}{2v \cdot Q}\ f(k_+)^{QCD},$$

where $\Delta s_{\mu\nu}$ is defined as the part of the spin structure not proportional to $s_{\mu\nu}$. The tensor $\Delta s_{\mu\nu}$ represents the residual spin effects not described by the EFT. The tensor $T_{\mu\nu}^{QCD}$ can be computed using the formula

$$T_{\mu\nu}^{QCD} = \frac{-i}{2}\ \mathrm{Tr}\ [P_+ \mathcal{M}_{\mu\nu}]$$

where $\mathcal{M}_{\mu\nu}$ is the Feynman amplitude for the forward scattering

$$b(v) + \gamma^*(q) \rightarrow b(v) + \gamma^*(q)$$

with the external spinors and the photon polarizations amputated.

3

3 Matching

The shape function receives perturbative QCD corrections which determine its evolution through a renormalization group equation[6, 2, 7]. The starting point of the evolution (boundary value) is determined by matching EFT onto full QCD. The matching constant (or coefficient function) is defined through the relation

$$f(k_+)^{QCD} = Z \; f(k_+)^{EFT} \qquad (9)$$

in which both the full QCD and the EFT shape functions are computed in perturbation theory up to a prescribed accuracy [2]. It is easier to compute the matching constant through the relation

$$F(k_+)^{QCD} = Z \; F(k_+)^{EFT}. \qquad (10)$$

By using relation (7) and the fact that Z is real (and positive), since it is a ratio of physical rates, one can indeed derive Eq.(9). At tree level:

$$F(k_+)^{QCD} = F(k_+)^{EFT} = \frac{1}{-k_+ + i\eta}$$

implying $Z = 1$ as it should. Throughout the paper, we will consider one-loop corrections in the double logarithmic approximation (DLA). In covariant gauges, the leading contributions come from the vertex correction diagrams (Fig.1); we choose the Feynman gauge for simplicity. The computation of the light-cone function gives in QCD (Fig.1a))

$$F(k_+)^{QCD} = \frac{1}{-k_+ + i\eta} \left(-\frac{1}{2}\right) \; a \; \log^2 \left(\frac{2\,m}{k_+ - i\eta}\right), \qquad (11)$$

where m is the b quark mass, and $a \equiv \alpha_s C_F / \pi$ [3].

In view of lattice-QCD applications, it is interesting to compute the matching constant of the shape function $f(k_+)$ in the EFT regularized by a hard cut-off. A regularization with a hard cut-off Λ (HC) on the spatial loop momenta is chosen

$$|\overrightarrow{l}| < \Lambda,$$

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[2] In perturbation theory we are actually computing the matrix elements with external heavy quarks states; for example, the right-hand side of Eq.(2) is replaced by $\langle b(v) \mid h_v^\dagger \delta(k_+ - iD_+) h_v \mid b(v) \rangle$.

[3] Balzereit et al.[2] have a similar result, but with m replaced by μ in Eq.(11).

while the loop energy l_0 varies on the entire real axis

$$-\infty < l_0 < +\infty.$$

We could consider ordinary lattice QCD regularization as well, such as for example the Wilson action; we expect similar results to hold in DLA [8].

In the EFT we have (Fig.1b))

$$F(k_+)^{EFT} = \frac{1}{-k_+ + i\eta} \left(-\frac{1}{2}\right) \; a \; \log^2 \left(\frac{2\,\Lambda}{k_+ - i\eta}\right).\qquad(12)$$

We have taken the external states in the effective theory on-shell because $k_+ \neq 0$ regulates the infrared divergences (soft and collinear ones) [4].

We explicitly see that the light-cone function (or equivalently the shape function) is ultraviolet divergent in the EFT, the divergence being a double logarithm of the cut-off, while it is finite in QCD (cfr. Eq.(11)). Inserting these expressions into Eq.(10) we find

$$
\begin{aligned}
Z &= 1 - \frac{1}{2}a \left[\log^2 \left(\frac{2\,m}{k_+ - i\eta}\right) - \log^2 \left(\frac{2\,\Lambda}{k_+ - i\eta}\right)\right] \\
&= 1 - a \log \left(\frac{m}{\Lambda}\right) \log \left(\frac{2\sqrt{m\,\Lambda}}{k_+ - i\eta}\right).
\end{aligned}
\qquad(13)
$$

The double logarithm of k_+, i.e. the leading infrared singularity, cancels in the difference of the integrals, implying the factorization of infrared physics into the effective theory light-cone function. In particular, if we choose

$$\Lambda = m \qquad (14)$$

as the matching scale, we get

$$Z = 1 \qquad (15)$$

also at one loop in DLA. We expect the physics to be independent from the ultraviolet regulator, as long as it is large enough; therefore, we can change the cut-off from the value (14) to any other one with an evolution equation in Λ. For example, in DLA the evolution equation leads to the well-known exponentiation of the one-loop amplitude [9]. Using our results:

$$F(k_+)^{EFT} = \frac{1}{-k_+ + i\eta} \exp\left[-\frac{1}{2} \; a \; \log^2 \left(\frac{2\,\Lambda}{k_+ - i\eta}\right)\right]$$

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[4] For further details of the derivation see [7].

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and the QCD light-cone function is

$$
\begin{aligned}
F(k_+)^{QCD} &= \frac{1}{-k_+ + i\eta} \left\{ 1 - \frac{1}{2}a \left[\log^2 \left(\frac{2\,m}{k_+ - i\eta} \right) - \log^2 \left(\frac{2\,\Lambda}{k_+ - i\eta} \right) \right] \right\} \\
&\quad \exp \left[-\frac{1}{2}\,a\,\log^2 \left(\frac{2\,\Lambda}{k_+ - i\eta} \right) \right].
\end{aligned}
\tag{16}
$$

With the choice (14), we have the QCD light-cone function with all large double logarithms exponentiated

$$
F(k_+)^{QCD} = \frac{1}{-k_+ + i\eta} \exp \left[-\frac{1}{2}\,a\,\log^2 \left(\frac{2\,m}{k_+ - i\eta} \right) \right].
$$

In general, the effective theory resums the $\alpha^n \log^k m/k_+$ terms in the form of renormalization-group (ultraviolet) logarithms $\alpha^n \log^k \Lambda/k_+$ [7].

Up to now we have considered the standard EFT. It is interesting to study a different effective field theory (EFT') where the b quark is treated in full QCD while the light quark is still taken in the LEET[5][10]. This effective theory contains corrections of the form k/m while it neglects corrections of the form $k/v \cdot Q$. In this EFT' theory we have computed the light-cone function $(\Lambda \gg m)$

$$
\begin{aligned}
F(k_+)^{EFT'} &= \frac{1}{-k_+ + i\eta} \left(-\frac{1}{2} \right) a \left[\log^2 \left(\frac{2\,\Lambda}{k_+ - i\eta} \right) - \log^2 \left(\frac{2\,\Lambda}{m} \right) \right] \tag{17} \\
&= \frac{1}{-k_+ + i\eta} \left(-\frac{1}{2} \right) a\,\log \left(\frac{m}{k_+ - i\eta} \right) \log \left(\frac{4\,\Lambda^2}{m(k_+ - i\eta)} \right).
\end{aligned}
$$

The result is still ultraviolet divergent, as in standard EFT, but the divergence is reduced to a single logarithmic one. This result is in contrast to naive power counting; the addition of the quadratic term in the heavy quark propagator, while reducing the singularity, does not completely eliminate the divergences of the effective theory. It is also interesting to note that the double logarithm of k_+ has the same coefficient as in QCD or in the ordinary EFT. In the EFT', the matching constant reads:

$$
Z' = 1 - a\,\log \frac{m}{\Lambda}\,\log \left(\frac{m}{k_+ - i\eta} \right).
$$

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[5]In DLA this is equivalent to replace the HQET field in Eq.(2) with a scalar field of mass m.

As in the usual EFT, $Z = 1$ at $\Lambda = m$ (cfr. Eq.(15)).

Let us now go back to standard effective field theory (EFT), but consider this time another regularization, i.e. dimensional regularization. The bare light-cone function at fixed ϵ is given by (Fig.1b))

$$
\begin{aligned}
F(k_+)^{EFT} &= \frac{1}{-k_+ + i\eta} \left(-\frac{a}{2} \right) \frac{\Gamma(1+\epsilon)\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{\epsilon^2} \left(\frac{\mu}{k_+ - i\eta} \right)^{2\epsilon} (18) \\
&= \frac{1}{-k_+ + i\eta}\, a \left[-\frac{1}{2\epsilon^2} - \frac{1}{\epsilon} \log \left(\frac{\mu}{k_+ - i\eta} \right) - \log^2 \left(\frac{\mu}{k_+ - i\eta} \right) \right]
\end{aligned}
$$

where $\epsilon \equiv 2 - D/2$, D is the space-time dimension and μ is the regularization scale [6]. The above formula is in agreement with Balzereit et al.[2]. The double and the simple pole are of ultraviolet nature; the finite term containing the $\log^2 k_+$ has an additional factor two with respect to the HC result (Eq.(12)) or the QCD result (Eq.(11)). Computing the coefficient function according to Eq.(10) we have

$$
Z = 1 + a \left[\frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \log \left(\frac{\mu}{k_+ - i\eta} \right) - \frac{1}{2} \log^2 \left(\frac{2\,m}{k_+ - i\eta} \right) + \log^2 \left(\frac{\mu}{k_+ - i\eta} \right) \right].
$$

Unlike what happens with the HC regularization, the matching constant above contains a double logarithm of k_+. This term does not drop, no matter what the choice of the matching scale is; for instance, taking

$$
\mu = 2\,m
$$

we have

$$
Z\,(\mu = 2m) = 1 + a \left[\frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \log \left(\frac{2\,m}{k_+ - i\eta} \right) + \frac{1}{2} \log^2 \left(\frac{2\,m}{k_+ - i\eta} \right) \right].
$$

This result has to be compared with the result (13) in the HC regularization, where the leading infrared singularity cancels in the matching at any value of Λ.

Let us trace the technical origin of the difference in the factor two between the two regularizations. With the natural correspondence

$$
\log \frac{2\Lambda}{\mu} \leftrightarrow \frac{1}{\epsilon}
$$

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[6] Actually, in the last member of Eq. (18), μ^2 should be replaced by $\mu^2 4\pi \exp[-\gamma_E]$ (γ_E is the Euler constant), even though this rescaling does not affect DLA results.

the HC result has the structure

$$\log^2\left(\frac{2\,\Lambda}{k_+}\right) \;\leftrightarrow\; \frac{1}{\epsilon^2}\left[1 + \epsilon \log\left(\frac{\mu}{k_+}\right)\right]^2 = \frac{1}{\epsilon^2} + \frac{2}{\epsilon}\log\left(\frac{\mu}{k_+}\right) + \log^2\left(\frac{\mu}{k_+}\right),$$

while the DR result has the structure

$$\frac{1}{\epsilon^2}\left(\frac{\mu}{k_+}\right)^{2\epsilon} = \frac{1}{\epsilon^2}\exp\left[2\epsilon\log\frac{\mu}{k_+}\right] = \frac{1}{\epsilon^2} + \frac{2}{\epsilon}\log\left(\frac{\mu}{k_+}\right) + 2\log^2\left(\frac{\mu}{k_+}\right).$$

The HC regularization involves a power function, while DR involves an exponential function. Besides the first two terms (the double pole and the simple one), these two functions do not coincide, the difference being in fact a factor two in the finite parts.

In QCD the quantity k_+ is an (infrared) kinematical variable (see (3)), i.e. it can be varied changing the external states. In the EFT the quantity k_+ is instead an index labelling the operators entering the definition of the shape-function (2) [2, 3]. This index is continuos and has the dimension of a mass. It is natural to render k_+ adimensional taking the ratio

$$\frac{k_+}{m}\;.$$

In the EFT, it is allowed to change the regularization scheme according to a ultraviolet finite replacement of the form

$$\frac{1}{\epsilon} \to \frac{1}{\epsilon} + \log\frac{k_+}{2\,m} \tag{19}$$

which renders the coefficient of the double logarithm of k_+ equal to the QCD one:

$$\begin{aligned}
\frac{1}{\epsilon^2}\left(\frac{\mu}{k_+}\right)^{2\epsilon} &= \frac{1}{\epsilon^2} + \frac{2}{\epsilon}\log\frac{\mu}{k_+} + 2\log^2\frac{\mu}{k_+} \\
&\to \frac{1}{\epsilon^2} + \frac{2}{\epsilon}\log\frac{\mu}{2m} + \log^2\frac{2m}{k_+} + 2\log\frac{\mu}{k_+}\log\frac{\mu}{2m}
\end{aligned} \tag{20}$$

The simple pole in Eq.(20) turns out to have a coefficient which is independent of k_+. The replacement (19) is equivalent to a finite change of

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renormalization prescription. At $\mu = 2\,m$, the right hand side of Eq.(20) takes the particularly simple form

$$\frac{1}{\epsilon^2} + \log^2 \frac{2\,m}{k_+}.$$

The matching constant reads in the new non-minimal dimensional scheme

$$Z = 1 + a \left[\frac{1}{2\,\epsilon^2} + \frac{1}{\epsilon} \log \left(\frac{\mu}{2\,m} \right) + \log \left(\frac{\mu}{2\,m} \right) \log \left(\frac{\mu}{k_+ - i\eta} \right) \right].$$

Taking $\mu = 2m$ results in a matching constant containing only the double (local) pole:

$$Z(\mu = 2m) = 1 + a\,\frac{1}{2\,\epsilon^2}.$$

The derivative with respect to $\log k_+$ of the last member in Eq.(20) (that controls the evolution [2, 6, 7]) reads:

$$-2 \log \frac{\mu}{k_+}.$$

The simple pole cancels as well as the dependence on the heavy quark mass.

4   Conclusions

We have shown that the shape function can be consistently matched onto QCD both with a hard cutoff and with dimensional regularization. In the former case the matching is very simple and involves only the difference of the quantum corrections in the two theories. In dimensional regularization a consistent matching requires to go to the non-minimal scheme specified by Eq.(19).

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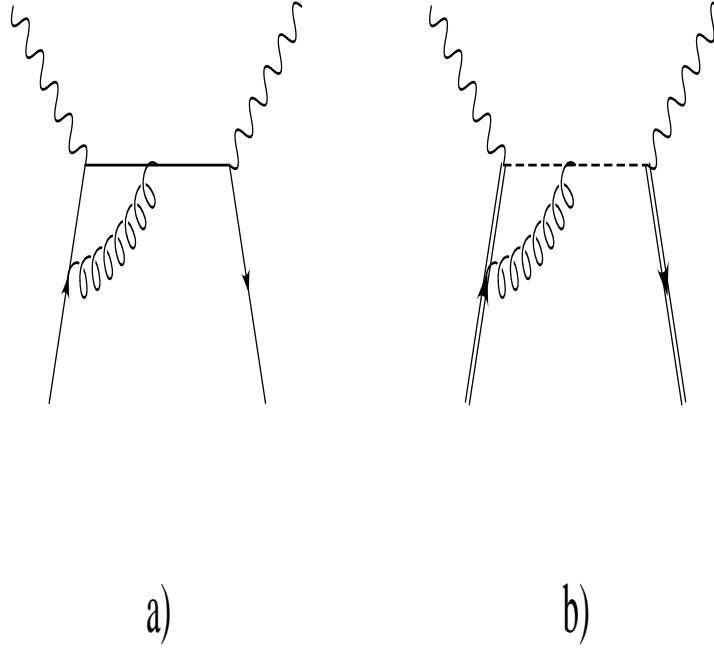
Figure 1: *Vertex corrections to the light cone function $F(k_+)$ in QCD (a) and in the EFT (b). The external waved lines are photon lines. The plain line in a) denotes a massive or a massless fermion propagator; the gluon line connects the heavy and the light quarks. In b), the double line denotes the heavy quark in the HQET, while the dashed line represents the light propagator in the LEET. Both diagrams have multiplicity 2.*